The relation between $\delta E(t_k+)$ and $\delta E(t_{k+1}-)$ is derived in Appendix B, and allows us to write

$$\delta E(t_{k+1}-) = N_{k+1}\delta t_k \tag{14}$$

With Eqs (12) and (14) and the recurrence formulas, Eqs (5) and (7), we can now evaluate the matrices M_i and N_i ($i > k+1$), which enable us to write

$$\delta X(t_i-) = M_i\delta t_k \qquad (i > k) \tag{15}$$

$$\delta E(t_i-) = N_i\delta t_k \qquad (i > k) \tag{16}$$

The matrices M_k and N_k are just the time derivatives of X and E at t_k, as determined from Eqs (1) and (2)

It should be noted that it is not necessary to solve additional differential equations in order to apply this procedure The matrices used in the recurrence formulas (5) and (7) are all quantities which presumably have already been computed for the nominal trajectory This is due principally to the fact that the vehicle follows an undisturbed free fall trajectory between corrections, which results in the uncoupling of Eq (A4) from Eq (A3) If Eq (A4) were not homogeneous, we would be forced to compute the transition matrix θ by integration of Eqs (A3) and (A4)

The analysis just given applies only if measurements are made continuously In the case of discrete measurements, Eq (2) would not be applicable, but would be replaced by Eq (7 1), and the remainder of the derivation would be altered accordingly

Appendix A: Perturbed Riccati Equation

The solution of Eq (2) is given by Kalman[2] as

$$E = [\theta_{21} + \theta_{22}E_0][\theta_{11} + \theta_{12}E_0]^{-1} \tag{A1}$$

E_0 and E here will represent values at the beginning and end of an interval between corrections, t_{i+} and t_{i+1-} The matrix

$$\theta = \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{21} & \theta_{22} \end{bmatrix} \tag{A2}$$

is the transition matrix of

$$\dot{\mathbf{x}} = -F^T\mathbf{x} + H^TR^{-1}H\mathbf{w} \tag{A3}$$

$$\dot{\mathbf{w}} = F\mathbf{w} \tag{A4}$$

for the time interval in question

Obviously, $\theta_{21} = 0$ in this case, and $\theta_{22} = \Phi_{i+1\ i}$; θ_{11} is the transition matrix of the Eq (A3), and can be shown to be

$$\theta_{11} = \Phi_{i+1\ i}{}^{-T} \tag{A5}$$

From Eq (A1) we can deduce that, with $\theta_{21} = 0$,

$$\theta_{12} = E^{-1}\theta_{22} - \theta_{11}E_0^{-1} \tag{A6}$$

By perturbing Eq (A1) we find

$$\begin{aligned}\delta E &= [\theta_{22}\delta E_0][\theta_{11} + \theta_{12}E_0]^{-1} - \\ &\quad [\theta_{21} + \theta_{22}E_0][\theta_{11} + \theta_{12}E_0]^{-1}\theta_{12}\delta E_0[\theta_{11} + \theta_{12}E_0]^{-1} \\ &= [\theta_{22} - E\theta_{12}]\delta E_0[\theta_{11} + \theta_{12}E_0]^{-1}\end{aligned} \tag{A7}$$

With $\theta_{21} = 0$, this can be written, using Eq (A1),

$$\delta E = [\theta_{22} - E\theta_{12}]\delta E_0[\theta_{22}E_0]^{-1}E \tag{A8}$$

which, together with Eq (A6), gives

$$\delta E = [E\Phi^{-T}E_0^{-1}]\delta E_0[E_0^{-1}\Phi^{-1}E] \tag{A9}$$

Appendix B: Perturbed Riccati Equation with Variable Initial Time

We now consider the effect of varying the initial time on Eq (A9) We consider only the special case $\theta_{21} = 0$ E_0 and E represent E_{k+} and E_{k+1-} Equation (A1) can be written

$$E = \Phi E_0[\Phi^{-T} + \theta_{12}E_0]^{-1} \tag{B1}$$

and Eq (A6) can be written

$$\theta_{12} = E^{-1}\Phi - \Phi^{-T}E_0^{-1} \tag{B2}$$

Hence, from Eq (B1),

$$\begin{aligned}\delta E &= [\delta\Phi E_0 + \Phi\delta E_0]\,[\Phi^{-T} + \theta_{12}E_0]^{-1} - \\ &\quad E[\delta\Phi^{-T} + \delta\theta_{12}E_0 + \theta_{12}\delta E_0][\Phi^{-T} + \theta_{12}E_0]^{-1} \\ &= [\delta\Phi E_0 + \Phi\delta E_0]E_0^{-1}\Phi^{-1}E - \\ &\quad E[\delta\Phi^{-T} + \delta\theta_{12}E_0 + \theta_{12}\delta E_0]\,E_0^{-1}\Phi^{-1}E\end{aligned} \tag{B3}$$

where, using Eq (11),

$$\delta\Phi = -\Phi F_k\delta t_k \tag{B4}$$

$$\delta\Phi^{-T} = -\Phi^{-T}\delta\Phi^T\Phi^{-T} = \Phi^{-T}F_k{}^T\delta t_k \tag{B5}$$

The quantity $\delta\theta_{12}$ is determined by noting that the solution of Eqs (A3) and (A4) can be expressed as

$$\mathbf{w}(t) = \Phi(t,t_k)\mathbf{w}(t_k) \tag{B6}$$

$$\mathbf{x}(t) = \Phi^{-T}(t,t_k)\mathbf{x}(t_k) + \int_{t_k}^{t} \Phi^{-T}(t,\tau)H^T(\tau)R^{-1}(\tau)H(\tau)\mathbf{w}(\tau)d\tau \tag{B7}$$

Since the last term represents $\theta_{12}(t,t_k)\mathbf{w}(t_k)$, it follows that

$$\theta_{12}(t,t_k) = \int_{t_k}^{t} \Phi^{-T}(t,\tau)H^T(\tau)R^{-1}(\tau)H(\tau)\Phi(\tau,t_k)d\tau \tag{B8}$$

from which

$$\begin{aligned}\frac{\partial}{\partial t_k}\theta_{12}(t_{k+1},t_k) &= -\Phi^{-T}H_k{}^TR_k{}^{-1}H_k + \int_{t_k}^{t_{k+1}} \Phi^{-T}(t_{k+1},\tau) \times \\ &\qquad H^T(\tau)R^{-1}(\tau)H(\tau)\frac{\partial}{\partial t_k}\Phi(\tau,t_k)d\tau \\ &= -\Phi^{-T}H_k{}^TR_k{}^{-1}H_k - \theta_{12}(t_{k+1},t_k)F_k\end{aligned} \tag{B9}$$

where we have made use of Eqs (B4) and (B8) in evaluating the last term

Combining Eqs (B2–B5 and B9), and simplifying,

$$\begin{aligned}\delta E &= E\Phi^{-T}[H_k{}^TR_k{}^{-1}H_k - F_k{}^TE_0{}^{-1} - E_0{}^{-1}F_k]\Phi^{-1}E\delta t_k + \\ &\qquad [E\Phi^{-T}E_0{}^{-1}]\delta E_0[E_0{}^{-1}\Phi^{-1}E]\end{aligned} \tag{B10}$$

References

[1] Denham W F and Speyer J L, Optimal measurement and velocity correction programs for midcourse guidance AIAA J 2, 896–907 (1964)

[2] Kalman R E and Bucy R S, 'New results in linear filtering and prediction theory J Basic Eng Trans Am Soc Mech Engrs 83D, 95–108 (March 1961)

[3] Battin R H, A statistical optimizing navigation procedure for space flight, ARS J 32, 1681–1696 (1962)

Comment on "Design Analysis of Earth-Lunar Trajectories: Launch and Transfer Characteristics"

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THE subject paper[1] includes a most informative graphical presentation and interpretation of the launch and transfer characteristics of earth-lunar trajectories Declination

Received November 22, 1963

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(δ_s) as a function of right ascension difference ($\alpha_L - \alpha$) for lines of constant launch azimuth (A_L) and constant central angle (Φ) are presented in Figs 2, 3, 7, and 8 of Ref 1 The object of this comment is to elaborate upon the subject paper by deriving the equations with which the lines of constant launch azimuth and constant central angle can be generated directly

In the paper the author begins with expressions for the components of the unit vectors toward the launch point at launch **L** and toward the moon at arrival **S** in terms of their respective right ascensions and declinations As in the paper, the components are

$$S_x = \cos\alpha \ \cos\delta \tag{1}$$

$$S_y = \sin\alpha \ \cos\delta \tag{2}$$

$$S \ = \sin\delta \tag{3}$$

$$L_x = \cos\alpha_L \cos\delta_L \tag{4}$$

$$L_y = \sin\alpha_L \cos\delta_L \tag{5}$$

$$L \ = \sin\delta_L \tag{6}$$

where

$$\mathbf{L} = iL_x + jL_y + kL \tag{7}$$

$$\mathbf{S} = iS_x + jS_y + kS \tag{8}$$

The author then derives an expression for Φ (the angle between **L** and **S** < 180°), obtaining

$$\cos\Phi = L_xS_x + L_yS_y + L\ S \tag{9}$$

He also derives an expression for the sine of the launch azimuth A_L:

$$\sin A_L = [(L_xS_y - L_yS_x)/\sin\Phi \cos\delta_L] \tag{10}$$

In addition, he presents an expression for the tangent of A_L:

$$\tan A_L = \frac{\sin\alpha_L S_x - \cos\alpha_L S_y}{(\cos\alpha_L S_x + \sin\alpha_L S_y)\sin\delta_L - \cos\delta_L S_x} \tag{11}$$

which, with the sine A_L, defines the quadrant of A_L

He then states that the preceding equations are used to draw the curves of Figs 2, 3, 7, and 8 However, there is a significant amount of mathematical expansion necessary to develop the expressions which are used to draw the curves

If we carry the development of the foregoing equations further, we can see from substitution that Eq (9) becomes

$$\cos\Phi = \cos\alpha_L \cos\delta_L \cos\alpha \ \cos\delta \ + \sin_L \cos\delta_L \sin\alpha \ \cos\delta \ + \sin\delta_L \sin\delta \tag{12}$$

or, factoring,

$$\cos\Phi = \cos\delta_L \cos\delta_L(\cos\alpha_L \cos\alpha \ + \sin\alpha_L \sin\alpha\ ) + \sin\delta_L \sin\delta \tag{13}$$

which, by a trigometric identity, gives

$$\cos\Phi = \cos\delta_L \cos\delta \ \cos(\alpha_L - \alpha\ ) + \sin\delta_L \sin\delta \tag{14}$$

Again by substitution, Eq (10) becomes

$$\sin A_L = \frac{\cos\alpha_L \cos\delta_L \sin\delta_s \sin\alpha \ - \sin\alpha_I \cos\delta_I \cos\alpha \ \cos\delta_s}{\sin\Phi \cos\delta_L} \tag{15}$$

or

$$\sin A_L = \frac{\cos\delta_L \cos\delta_s(\cos\alpha_L \sin\alpha \ - \sin\alpha_L \cos\alpha\ )}{\sin\Phi \cos\delta_L} \tag{16}$$

and, finally,

$$\sin A_L = \frac{-\cos\delta \ \sin(\alpha_L - \alpha\ )}{\sin\Phi} \tag{17}$$

Equation (11) becomes

$$\tan A_L = \frac{\sin\alpha_L \cos\alpha_s \cos\delta_s - \cos\alpha_L \sin\alpha_s \cos\delta_s}{(\cos\alpha_L \cos\alpha \ \cos\delta \ + \sin\alpha_L \sin\alpha \ \cos\delta\ )\sin\delta_L - \cos\delta_L \cos\delta_s} \tag{18}$$

$$\tan A_L = \frac{\cos\delta_s(\sin\alpha_L \cos\alpha \ - \cos\alpha_L \sin\alpha_L)}{(\cos\alpha_L \cos\alpha_s + \sin\alpha_L \sin\alpha\ ) \sin\delta_L \cos\delta_s - \cos\delta_L \cos\delta} \tag{19}$$

or

$$\tan A_L = \frac{\sin(\alpha_L - \alpha_s)}{\sin\delta_L \cos(\alpha_L - \alpha\ ) - \cos\delta_L} \tag{20}$$

Equations (14, 17, and 20) now define Φ and A_L in terms of the difference in right ascension

Instead of using Eq (20) to determine the quadrant of A_L, an alternate method will be used in this discussion The alternate method does not depend on the Kohlase equation [Eq (11), see subject paper's Ref 6]

The new approach uses the **W** unit vector defined in the paper as

$$\mathbf{W} = \mathbf{L} \times \mathbf{S}/|\mathbf{L} \times \mathbf{S}| \tag{21}$$

Obviously the **W** vector is normal to the plane defined by the **S** and **L** vectors If a new vector **Q** is defined as

$$\mathbf{Q} = \mathbf{W} \times \mathbf{L} \tag{22}$$

the vector **Q** will be perpendicular to **L** and in the plane defined by **L** and **S** If the **Q** vector is directed above the equatorial plane (i e , Q is positive), A_L will be in the first or fourth quadrant and the cosine of A_L will be positive If **Q** is directed below the equatorial plane (Q is negative), A_L will be in the second or third quadrant and the cosine of A_L will be negative The cosine of A_L can then be expressed as

$$\cos A_L = (Q\ /|Q_z|)(1 - \sin^2 A_L)^{1/2} \tag{23}$$

Q also can be written as a function of the right ascensions difference

$$Q \ = W_xL_y - L_x\ W_y \tag{24}$$

or

$$Q \ = \frac{(L_yS_z - S_yL\ )L_y}{\sin\Phi} - \frac{(L_zS_x \ - S_zL_x)L_x}{\sin\Phi} \tag{25}$$

$$Q \ \sin\Phi = L_y{}^2S \ - S_yL_zL_y - L\ S_xL_x + S\ L_x{}^2 \tag{26}$$

$$Q \ \sin\Phi = \sin^2\alpha_L \cos^2\delta_L \sin\delta \ - \sin\alpha \ \cos\delta \ \sin\delta_L \sin\alpha_L \cos\delta_L - \sin\delta_L \cos\alpha \ \cos\delta \ \cos\alpha_L \cos\delta_L + \sin\delta \ \cos^2\alpha_L \cos^2\delta_L \tag{27}$$

$$Q_z \sin\Phi = \sin\delta \ \cos^2\delta_L(\sin^2\alpha_L + \cos^2\alpha_L) - \cos\delta \ \sin\delta_L \cos\delta_L(\sin^2\alpha_s \sin\alpha_L + \cos\alpha \ \cos\alpha_L) \tag{28}$$

or

$$Q \ = \frac{\sin\delta_s \cos^2\delta_L - \cos\delta_s \sin\delta_L \cos\delta_L \cos(\alpha_L - \alpha_s)}{\sin\Phi} \tag{29}$$

Equation (23) along with Eq (17) can be used to determine A_L At this point the equation for the curves of constant Φ for Figs 2, 3, 7, and 8 can be written

Equation (14) can be rewritten as

$$\cos(\alpha_L - \alpha\ ) = \frac{\cos\Phi - \sin\delta_L \sin\delta_s}{\cos\delta_L \cos\delta} \tag{30}$$

By choosing a Φ and incrementing δ from $-90°$ to $+90°$, a curve of constant Φ can be computed from Eq (30) It should be noted that, for every value of δ used in Eq (30), two values of $(\alpha_L - \alpha)$ will be obtained Both points lie on the same Φ curve, however, and so there is no problem in computing the curve

The curves of constant A_L are not obtained so directly From Eq (17),

$$\sin A_L = -\frac{\cos\delta_s \sin(\alpha_L - \alpha)}{(1 - \cos^2\Phi)^{1/2}} \tag{31}$$

since

$$\cos^2\Phi + \sin^2\Phi = 1 \tag{32}$$

or

$$(1 - \cos^2\Phi)^{1/2} = -\frac{\cos\delta_s \sin(\alpha_L - \alpha)}{\sin A_L} \tag{33}$$

Squaring,

$$1 - \cos^2\Phi = \frac{\cos^2\delta \ \sin^2(\alpha_L - \alpha)}{\sin^2 A_L} \tag{34}$$

From Eq (14),

$$1 - (\cos\delta_L \cos\delta \ \cos(\alpha_L - \alpha) + \sin\delta_L \sin\delta)^2 = \cos^2\delta \ \sin^2(\alpha_L - \alpha)/\sin^2 A_L \tag{35}$$

Expanding

$$1 - (\cos^2\delta_L \cos^2\delta \ \cos^2(\alpha_L - \alpha) + 2\cos\delta_L \cos\delta \ \cos(\alpha_L - \alpha)\sin\delta_L \sin\delta + \sin^2\delta_L \sin^2\delta) = \cos^2\delta \ \sin^2(\alpha_L - \alpha)/\sin^2 A_L \tag{36}$$

$$\sin^2 A_L - \sin^2 A_L \cos^2\delta_L \cos^2\delta \ \cos^2(\alpha_L - \alpha) - 2\sin^2 A_L \cos\delta_L \cos\delta \ \cos(\alpha_L - \alpha) \times \sin\delta_L \sin\delta - \sin^2 A_L \sin^2\delta_L \times \sin^2\delta = \cos^2\delta\,[1 - \cos^2(\alpha_L - \alpha)] \tag{37}$$

$$= \cos^2\delta_s - \cos^2\delta \ \cos^2(\alpha_L - \alpha) \tag{38}$$

$$\cos^2(\alpha_L - \alpha_s)(\cos^2\delta_s - \sin^2 A_L \cos^2\delta_L \cos^2\delta) - \cos(\alpha_L - \alpha)(2\cos\delta_L \cos\delta \ \sin^2 A_L \sin\delta_L \sin\delta) + (\sin^2 A_L - \sin^2 A_L \sin^2\delta_L \sin^2\delta - \cos^2\delta) = 0 \tag{39}$$

Let

$$\begin{aligned} A &= \cos^2\delta - \sin^2 A_L \cos^2\delta_L \cos^2\delta \\ B &= -2\cos\delta_L \cos\delta \ \sin^2 A_L \sin\delta_L \sin\delta \\ C &= \sin^2 A_L - \sin^2 A_L \sin^2\delta_s - \cos^2\delta_s \end{aligned} \tag{40}$$

Then

$$\cos(\alpha_L - \alpha_s) = \frac{-B \pm (B^2 - 4AC)^{1/2}}{2A} \tag{41}$$

Equation (41) enables the calculation of points for the curves of constant A_L The points are obtained in the same manner as the points for constant Φ, that is, choosing A_L and incrementing δ from $-90°$ to $+90°$

A remaining problem is that multiple answers are obtained in the A_L case as well as in the constant Φ case However, in the constant A_L case, four values of $(\alpha_L - \alpha)$ are obtained for each δ increment, and only two of these correspond to the desired A_L The other two correspond to the explement of the A_L used in Eq (41) To identify which $(\alpha_L - \alpha)$ go with the desired A_L and which to the explement of A_L, it is necessary to substitute each $(\alpha_L - \alpha)$ obtained from Eq (41) into Eq (14), obtaining

$$\cos\Phi = \cos\delta_L \cos\delta_s \cos(\alpha_L - \alpha) + \sin\delta_L \sin\delta \tag{42}$$

Then using Eq (17) and the Φ obtained in Eq (42), compute

$$\sin A_L = -\frac{\cos\delta \ \sin(\alpha_L - \alpha)_i}{\sin\Phi} \tag{43}$$

and obtain the $\cos A_L$ from Eq (23), still using $(\alpha_L - \alpha)$ and the Φ from Eq (42):

$$\cos A_L = (Q/|Q|)(1 - \sin^2 A_L)^{1/2} \tag{44}$$

From Eqs (43) and (44), A_L and its correct quadrant can be determined Each $(\alpha_L - \alpha_s)$ can be matched with A_L or the explement of A_L for identification of the curve

It is now possible to compute and identify in a straightforward manner all of the points necessary to draw the curves in Figs 2, 3, 7, and 8 Using these equations, it now appears that much of the remaining problem of earth-lunar trajectory design as outlined in the paper can be mechanized However, as it was the primary object of this comment to elaborate upon the subject paper by deriving the equations from which the lines of constant launch azimuth and constant central angle can be generated, the problem of mechanization of the trajectory design problem will not be pursued further

Reference

[1] Michaels, J E, "Design analysis of earth-lunar trajectories: launch and transfer characteristics," AIAA J 1, 1342–1350 (1963)

Comments on Exhaust Flow Field and Surface Impingement

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Nomenclature

k = $\gamma(\gamma - 1)M_e^2$
h = distance along streamline from point source (nozzle exit)
r = radial coordinate normal to axis
r_e = nozzle exit radius
θ = streamline inclination angle relative to axis
M = Mach number
γ = ratio of ideal-gas specific heats
ρ = density
ρ_0 = nozzle stagnation density
ν = Prandtl Meyer turning angle
p_s = static pressure on surface beneath nozzle
p_{re} = normal shock recovery pressure at nozzle exit
p_0 = nozzle stagnation pressure

Subscripts

e = conditions at nozzle exit
a = conditions on axis

In recent months there has been interest in the problem of the impingement of a jet exhaust on simulated lunar surfaces The problem has been described both experimentally[1] and theoretically[2, 3] In the June issue of the AIAA Journal there appeared two Technical Notes[4, 5] that seemed to complicate unnecessarily the analysis of the free-jet expansion and the surface pressure distribution The purpose of the present comment is to recall briefly the simple results of the theory[3] and to make a more complete comparison with the experimental data of Ref 1

First, with regard to the expansion of the free jet into a vacuum, most of the mass and momentum of the jet are contained in a central core in which the density decreases both along and normal to the axis, and at large distances the density distribution should be independent of the opening angle of the nozzle Furthermore, one would not expect

Received September 27, 1963
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